

Expansive billiard flows

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Abstract

We consider polygonal billiards on surfaces of constant curvature. We show that the usual phase space of the billiard flow can be extended to a compact three dimensional manifold without boundary that is a circle bundle. We extend the dynamics C^∞ to the manifold defined. We relate the expansiveness of the billiard flow with the existence of periodic orbits and with the injectivity of the itinerary map for billiard tables in the hyperbolic disc, the Euclidean plane and the sphere.

1 Introduction

In the study of polygonal billiards of the Euclidean plane with rational angles (i.e. rational polygons) there is a standard construction given in [11]. By a finite unfolding of the rational polygon a compact surface without boundary is defined. Then, if we fix a direction, a complete flow is defined with one singular point associated to each vertex of the polygon. The singularities are of saddle type but may not be hyperbolic. This construction gives a natural way to see the billiard flow of a rational billiard in the setting of complete flows on compact surfaces.

Here we consider this problem for general polygonal billiards in the Euclidean plane, the hyperbolic disc and the sphere. First we analyze a neighborhood of the vertices in polar coordinates to show how to compactify the usual three dimensional phase space of the billiard flow. For each singular point we add a circle with two hyperbolic singularities. The dynamics on each of this circles is a South-North flow. The billiard flow obtained is complete and C^∞ . By construction the phase space is a compact three dimensional manifold without boundary and we show that it is a circle bundle.

We study the expansive properties of the billiard flow. Since there are singular points it is natural to consider the definition of expansiveness given in [8] (k^* -expansiveness). For expansive flows on three dimensional manifolds without singular points it is known, see [9], that the fundamental group of the phase space has exponential growth. Here we show that the billiard flow associated to a polygon in the hyperbolic disc is expansive but if the polygon is simply connected or homeomorphic to an annulus the phase space manifold has not exponential growth. It is not a contradiction because there are singular points in our case.

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In the subject of flat polygonal billiards an important open problem is to prove the existence or not of periodic orbits. In [3] it is shown that excluding the periodic orbits the collision map is expansive. Here we extend their ideas and show that a flat polygonal billiard has periodic orbits if and only if the billiard flow is expansive.

In the case of positive curvature, we show that if there are periodic orbits then the billiard flow can not be expansive. We also give an example of a polygon on the sphere without periodic orbits, but its billiard flow is not expansive.

2 Phase space

In that section we are going to construct the phase space. The billiard flow will be defined in the next section. Given a polygon D we will consider the double billiard surface S and compactifying its unit tangent bundle we will obtain our manifold M . Finally we study some topological properties of M .

We start giving basic definitions. Let S' be a C^∞ Riemannian manifold of dimension 2. We say that $D \subset S'$ is a *polygon* if D is compact, connected, $\partial D = \partial \text{int}(D)$ and its boundary is a finite union of geodesic arcs. Assume that $\partial D = \cup_{i=1}^n \gamma_i$ with γ_i being geodesic arcs and $\gamma_i \cap \gamma_j \subset (\partial \gamma_i) \cap (\partial \gamma_j)$. We denote by $\partial \gamma_i$ the end points of the arc. We say that the end points of γ_i are the *vertices* of the polygon. Denote by V' the set of vertices of D .

Consider D' an isometric disjoint copy of D and $\varphi: D \rightarrow D'$ an isometry. Let

$$S = \frac{D \cup D'}{\simeq}$$

where \simeq is the equivalence relation on $D \cup D'$ generated by $x \simeq \varphi(x)$ for all $x \in \partial D$. Denote by $[x]$ the class of $x \in D \cup D'$. With the quotient topology, S is a compact surface without boundary.

Definition 1. We say that S is the *double billiard surface* of the billiard D .

Example 1. If D is homeomorphic to a disc, then S is homeomorphic to a sphere. If D is homeomorphic to an annulus then S is topologically a torus.

Let $V = \{[x] : x \in V'\}$ denote the *vertices* or *singularities* of S . Consider $\phi_1: D \rightarrow S$ and $\phi_2: D' \rightarrow S$ given by $\phi_1(x) = [x]$ and $\phi_2(x) = [x]$. This maps are diffeomorphisms and they induce Riemannian metrics on their images. Since the boundary of D and D' are geodesics, we have that $S \setminus V$ admits a Riemannian metric that coincides with the induced by ϕ_1 and ϕ_2 . The points of V are singular points of the metric. Let $T^1(S \setminus V)$ denote the unit tangent bundle of $S \setminus V$. We have that $T^1(S \setminus V)$ is a noncompact three dimensional manifold. The task now is to give a compactification.

Denote by dist the distance in $T^1 S$ induced by the Riemannian metric in $S \setminus V$. Now fix $p \in V$ and consider $\varepsilon > 0$ small enough such that the ε -neighborhood of $p \in S$ is a disc. Define

$$U_p = \{(x, v) \in T^1(S \setminus V) : 0 < \text{dist}(x, p) < \varepsilon\}. \quad (1)$$

Now we will introduce polar coordinates. Consider the map $r: U_p \rightarrow (0, \varepsilon)$ given by $r(x, v) = \text{dist}(x, p)$. Suppose that $\gamma_1 = [\gamma'_1]$ and $\gamma_2 = [\gamma'_2]$ with γ'_1 and γ'_2 being the geodesic arcs in the polygon D meeting at p . Given $(x, v) \in U_p$ denote

by γ the angle between the geodesic segment from p to x and γ_1 . If θ is the angle between γ_1 and γ_2 , then γ is a map from U_p to $\mathbb{R}/2\theta\mathbb{Z}$. Let β be the angle between the vector $v \in T_x S$ and the segment from p to x . The map β goes from U_p to $\mathbb{R}/2\pi\mathbb{Z}$. See figure 1.

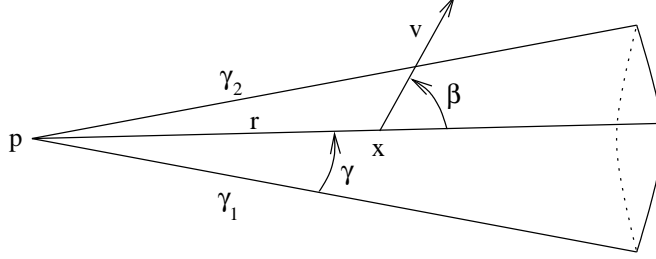


Figure 1: The coordinates (r, γ, β) near a singular point p .

Therefore, the map $\phi_p: U_p \rightarrow V_p = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < \varepsilon^2\} \times \mathbb{R}/2\pi\mathbb{Z}$ given by

$$\phi_p(x, v) = (r \cos(\gamma\pi/\theta), r \sin(\gamma\pi/\theta), \beta) \quad (2)$$

is a diffeomorphisms.

For each singular point p consider a circle $\gamma_p = \{p\} \times \mathbb{R}/2\pi\mathbb{Z}$. Extend ϕ_p to a map (that we still call ϕ_p) as

$$\phi_p: U_p \cup \gamma_p \rightarrow \{(x, y) \in \mathbb{R}^2 : 0 \leq x^2 + y^2 < \varepsilon^2\} \times \mathbb{R}/2\pi\mathbb{Z}$$

where $\phi_p(p, \beta) = ((0, 0), \beta)$ for $(p, \beta) \in \gamma_p$. Now procede in the same way on the other vertices. We obtain a compact three dimensional manifold that we call

$$M = T^1(S \setminus V) \cup_{p \in \text{Sing}} \gamma_p.$$

It will be called as the *phase space* of the billiard D . Now we will study the bundle structure of M .

Theorem 1. *The phase space M is a circle bundle. Moreover, if D is orientable then the bundle structure can be associated to an action of the circle.*

Proof. First notice that $T^1(S \setminus V)$ is diffeomorphic to the product $S \times \mathbb{S}^1$. It is because there exist a vector field in S with singular points in V . The bundle structure of $T^1(S \setminus V)$ extends to the singular points p in V according to the charts ϕ_p . So $\pi: M \rightarrow S$ defined by $\pi(x, v) = x$ if $x \notin V$ extends to M as $\pi(\gamma_p) = p$ for $p \in V$, and π is a submersion with circle fibers.

If D is orientable then the double billiard surface S is orientable too. Then given $\theta \in \mathbb{S}^1$, one defines the action on (x, v) , with $x \notin V$, as $\theta \cdot (x, v) = (x, R_\theta(v))$, where R_θ is the rotation of angle θ in $T_x S$, according to a fixed orientation of S . The action in the coordinates (x, y, β) is defined by $\theta \cdot (x, y, \beta) = (x, y, R_\theta(\beta))$. \square

Now we will calculate the fundamental group of the phase space M that will be denoted as $\pi_1(M)$. Consider a non-singular point $p_* \in S \setminus V$ and a vector field Y with the following properties:

1. the set of singularities of Y is $V \cup \{p_*\}$,
2. the index of Y at each singular point $p \in V$, equals 1,
3. if we define $Z = Y/\|Y\|$ as a vector field on $S \setminus \text{Sing}(Y)$ then on each U_p , $p \in V$, Z is constant.

The construction of such vector field is standard. Thus we can extend Z to $S \setminus \{p_*\}$ to a global section of the fiber bundle $M \setminus \gamma_{p_*} \rightarrow S \setminus \{p_*\}$. Assume that S is orientable. The fundamental group of $S_{p_*} = S \setminus \{p_*\}$ is free and consider $a_1, b_1, \dots, a_g, b_g$ generators of $\pi_1(S_{p_*})$, where g is the genus of S .

Define in M the following curves

$$\begin{aligned}\hat{a}_i(t) &= (a_i(t), Z(a_i(t))), \\ \hat{b}_i(t) &= (b_i(t), Z(b_i(t))).\end{aligned}$$

Denote by q the base point of the group $\pi_1(S_{p_*})$ and let $X(S)$ be the Euler characteristic of the surface. Consider γ_q as the element of $\pi_1(M)$ corresponding to the fiber at q and let $N = |V|$ be the number of singular points (or vertices of the polygon D).

Theorem 2. *The fundamental group of M is:*

$$\pi_1(M) = \frac{\langle \hat{a}_1, \hat{b}_1, \dots, \hat{a}_g, \hat{b}_g, \gamma_q \rangle}{[\hat{a}_1, \hat{b}_1] \dots [\hat{a}_g, \hat{b}_g] = \gamma_q^{X(S)-N} \text{ and } \gamma_q \text{ commutes},}$$

where $\langle \dots \rangle$ denotes the free group generated.

Proof. We will use Van Kampen's Theorem (see [5] Theorem 1.20). For this consider a small disc $D_* \subset S$ around p_* and let $U_* = T^1(U_*)$ be a neighborhood of γ_{p_*} . Let $U' = M \setminus \gamma_{p_*}$, where γ_{p_*} is the fiber of p_* . We assume that the base point of $\pi_1(M)$ is in $U_* \cap U'$. We have that $M = U_* \cup U'$. Since Z is a global section of U' we have that U' is a trivial fiber bundle. Thus U' is homeomorphic to the product $S_{p_*} \times \mathbb{S}^1$ and we have that $\pi_1(U') = \langle \hat{a}_1, \hat{b}_1, \dots, \hat{a}_g, \hat{b}_g \rangle \times \mathbb{Z}\gamma_q$. Since $U_* = D_* \times \mathbb{S}^1$ we have that $\pi_1(U_*) = \mathbb{Z}\gamma_q$. Let $c \subset D_*$ be a loop around p_* in S_{p_*} and define $\hat{c}(t) = (c(t), Z(c(t)))$ the corresponding loop in $U' \cap U_* \subset M$. Then $\pi_1(U' \cap U_*) = \mathbb{Z}\gamma_q \times \mathbb{Z}\hat{c}$. On one hand, if we view \hat{c} in the fundamental group of U_* we have the relation $\hat{c} = \gamma_q^{\text{ind}_Y(p_*)}$, where $\text{ind}_Y(p_*)$ is the index of Y at p_* . On the other hand, viewing \hat{c} in U' , we have that $[\hat{a}_1, \hat{b}_1] \dots [\hat{a}_g, \hat{b}_g] = \hat{c}$, that is because c is homotopic to $[a_1, b_1] \dots [a_g, b_g]$ in $S \setminus \{p_*\}$. Now recall that $X(S) = \text{ind}_Y(p_*) + \sum_{q \in V} \text{ind}_Y(q) = \text{ind}_Y(p_*) + N$ because the index of the singular points in V is 1 and N is the number of singular points. The commutativity of γ_q follows by the bundle structure. \square

Let G be a finitely generated group, as is $\pi_1(M)$ and consider the function $\Gamma(n)$ defined as the number of distinct group elements of word-length (according to a fixed finite generator of G) not greater than n . We say that G has *exponential growth* if the function $\Gamma(n)$ dominates Ae^{an} for some $a, A \in \mathbb{R}^+$. See [9, 10] for comments of equivalent definitions.

Corollary 1. *Let N be the number of vertices of a simply connected billiard table D . Then $\pi_1(M) = \mathbb{Z}_{N-2}$ and has not exponential growth.*

Proof. If D is simply connected, then D is homeomorphic to a disc and S is topologically a sphere. Thus $X(S) = 2$ and $g = 0$. Then by the previous Theorem $\pi_1(M) = \mathbb{Z}_{N-2}$. So the group is cyclic and its growth is linear. \square

Example 2. *If the billiard table is a triangle then the phase space M is the sphere S^3 . It follows because the fundamental group of M is trivial and also it can be seen directly.*

3 Billiard flow

In that section we will show that the billiard flow can be reparameterized, immersed in M and extended C^∞ for billiards with constant curvature. We divide the study according to the sign of the curvature, but the procedure seems to be generalizable to arbitrary metrics. First we give an expression for the geodesic flow near a singular point. Then we show how to extend the flow to M and finally we study the singular points obtained.

3.1 Motion equations

We will describe the motion equations according to the sign of the curvature of the billiard table.

3.1.1 Flat billiards

Suppose that S' is flat, let p be a singular point and consider the neighborhood U_p defined by (1). Now the geodesic flow of S can be seen as a non-complete flow in $U_p \setminus \gamma_p$. In the coordinates r, γ, β (see Figure 1) it has the following equations:

$$\begin{cases} r_t \cos \beta_t - r_0 \cos \beta_0 = t, \\ \gamma_t - \gamma_0 + \beta_t - \beta_0 = 0, \\ r_t^2 = r_0^2 + 2tr_0 \cos \beta_0 + t^2. \end{cases}$$

Where r_0, γ_0, β_0 is an arbitrary initial condition. This equations follows by an elemental trigonometric analysis on Figure 2.

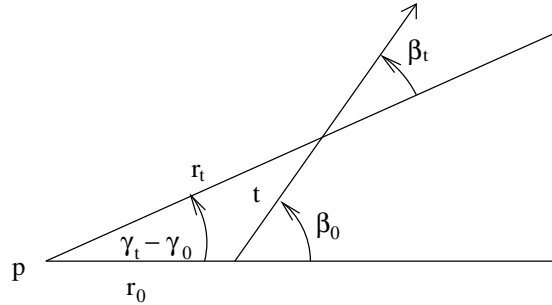


Figure 2: Geodesic flow near p .

Now differentiating in t and putting $t = 0$ we have:

$$\begin{cases} \dot{r} = \cos \beta, \\ \dot{\gamma} = \frac{1}{r} \sin \beta, \\ \dot{\beta} = -\frac{1}{r} \sin \beta. \end{cases}$$

These are the equations of the velocity field of the geodesic flow in the coordinates r, γ, β . Let

$$X_F(r, \gamma, \beta) = \left(\cos \beta, \frac{1}{r} \sin \beta, -\frac{1}{r} \sin \beta \right)$$

be the vector field in U_p given by the previous equations in the coordinates r, γ, β .

3.1.2 Negative curvature

Assume that S' has constant curvature -1 . In that case we use the Laws of sines and cosines of hyperbolic trigonometry, on the triangle of Figure 2, to conclude the following equations:

$$\begin{cases} \sin \beta_t \sinh t = \sin(\gamma_t - \gamma_0) \sinh r_0, \\ \sin \beta_t \sinh r_t = \sin(\pi - \beta_0) \sinh r_0, \\ \cosh r_t = \cosh r_0 \cosh t - \sinh r_0 \sinh t \cos(\pi - \beta_0). \end{cases}$$

Derivating in t and evaluating in $t = 0$ we have:

$$\begin{cases} \dot{r} = \cos \beta, \\ \dot{\gamma} = \frac{1}{\sinh r} \sin \beta, \\ \dot{\beta} = -\frac{\cosh r}{\sinh r} \sin \beta. \end{cases}$$

So we define the vector field

$$X_H(r, \gamma, \beta) = \left(\cos \beta, \frac{1}{\sinh r} \sin \beta, -\frac{\cosh r}{\sinh r} \sin \beta \right).$$

3.1.3 Positive curvature

Similar considerations, using spherical trigonometry, gives us:

$$\begin{cases} \sin \beta_t \sin t = \sin(\gamma_t - \gamma_0) \sin r_0, \\ \sin \beta_t \sin r_t = \sin(\pi - \beta_0) \sin r_0, \\ \cos r_t = \cos r_0 \cos t + \sin r_0 \sin t \cos(\pi - \beta_0). \end{cases}$$

Differentiating in t and evaluating in $t = 0$ we arrive to:

$$\begin{cases} \dot{r} = \cos \beta, \\ \dot{\gamma} = \frac{1}{\sin r} \sin \beta, \\ \dot{\beta} = -\frac{\cos r}{\sin r} \sin \beta \end{cases}$$

and then we consider

$$X_S(r, \gamma, \beta) = \left(\cos \beta, \frac{1}{\sin r} \sin \beta, -\frac{\cos r}{\sin r} \sin \beta \right).$$

3.2 Extension of the billiard flow

We start reparameterizing the billiard flow. Let $\rho_H, \rho_F, \rho_S: S \rightarrow \mathbb{R}$ be non-negative functions such that

$$\rho_*(x) = \begin{cases} \sinh \operatorname{dist}(x, p) & \text{if } * = H, \\ \operatorname{dist}(x, p) & \text{if } * = F, \\ \sin \operatorname{dist}(x, p) & \text{if } * = S, \end{cases}$$

for each $x \in U_p$. Let $Y_* = \rho_* X_*$ for $* = F, H$ or S . The vector field Y_* induces a reparametrized flow of the one given by X_* .

Now, using local charts, we will show that the flow extends smoothly to M . Consider the diffeomorphism

$$\psi: (0, \varepsilon) \times \mathbb{R}/2\theta\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow V_p = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < \varepsilon^2\} \times \mathbb{R}/2\pi\mathbb{Z},$$

given by

$$\psi(r, \gamma, \beta) = (r \cos(\gamma\pi/\theta), r \sin(\gamma\pi/\theta), \beta).$$

It is the diffeomorphism given in equation (2) expressed in the (r, γ, β) coordinates. Define $Z_* = d\psi Y_*$, a vector field in V_p . Let $\psi(r, \gamma, \beta) = (x, y, z)$. In the coordinates (x, y, z) the expression of Z_* is as follows:

$$Z_*(x, y, z) = \left(f_* x \cos z - \frac{\pi}{\theta} y \sin z, f_* y \cos z + \frac{\pi}{\theta} x \sin z, -f_* \sin z \right), \quad (3)$$

where $f_* = (1 - k(x^2 + y^2))^{1/2}$ with k denoting the curvature of the billiard table:

$$\begin{cases} k = 0 & \text{if } * = F, \\ k = -1 & \text{if } * = H, \\ k = 1 & \text{if } * = S. \end{cases}$$

The vector field Z_* extends and is C^∞ on $\operatorname{clos} V_p$. It corresponds with an extension of the vector field Y to $U_p \subset M$.

Now proceede in the same way on the other vertices, extending Y to the whole M . The billiard flow (in the usual sense) is immersed in M in a reparametrized way. We call *billiard flow* to the flow generated by Y . By construction it is C^∞ and it is complete because M is compact and without boundary.

3.3 Singularities of the billiard flow

By construction, the singular points of the billiard flow are in γ_p for $p \in V$. So we consider the coordinates (x, y, z) around p and the expression of the vector field given by (3). We have that

$$Z_*(0, 0, z) = (0, 0, -\sin z).$$

So on each γ_p there are two singular points: $(0, 0, 0)$ and $(0, 0, \pi)$. Differentiating one see that both singularities are hyperbolic. The point $(0, 0, 0)$ has a stable manifold of dimension 1 and an unstable manifold of dimension 2. The hyperbolic splitting of $(0, 0, \pi)$ has reversed dimensions.

4 Expansive billiard flows

In that section we study the expansive properties of the billiard flow defined by the vector field Y in the previous section.

4.1 Preliminaries

We start recalling some basic facts about expansive flows, the itinerary map of the billiard collision map and the unfolding technique. With this tools we will analyze the relationship of expansiveness, existence of periodic orbits and injectivity of the itinerary map for the 3 cases analyzed above: flat, hyperbolic and spherical.

4.1.1 Expansive flows

Expansiveness of a dynamical system is a stronger version of sensitivity of initial conditions and so it is related with chaotic systems. Examples of expansive flows are Anosov flows and the non-wandering set of a hiperbolic flow. The Lorenz attractor is an expansive flow with a singular point as shown in [8]. The expansive flows of compact surfaces are described in [1], essentially, as the suspension of minimal interval exchange maps. In Theorem 3 we show that there are singular expansive flows on some three dimensional manifolds.

Recall that a flow is said to be *expansive* if for all $\varepsilon > 0$ there exist an *expansive constant* $\delta > 0$ such that if $\text{dist}(\phi_{h(t)}(x), \phi_t(y)) < \delta$ for all $t \in \mathbb{R}$, being $h: \mathbb{R} \rightarrow \mathbb{R}$ an increasing homeomorphism with $h(0) = 0$, then x and y are contained in an orbit segment of diameter less than ε . As shown in [1], that definition is equivalent with the notion of k^* -expansiveness given in [8].

4.1.2 The itinerary map

The itinerary of a trjectory is simply the sequence of the sides of the polygon that it hits. Some technical difculties are due to the fact that a trajectory may hit a vertex.

Label the sides of the polygon D as $1, 2, \dots, N$. As is usual in the theory of billiards, we consider the collision map f associated to the polygon D (see [2]). Denote by $\gamma_1, \dots, \gamma_N$ the geodesic arcs in ∂D and define $\mathbb{Z}_{(x,v)}$ the maximal integer interval such that for all $n \in \mathbb{Z}_{(x,v)}$ if $(y, w) = f^n(x, v)$ then y is not a vertex. Let X be the set of all the functions $f: I \rightarrow \{1, 2, \dots, N\}$ where $I \subset \mathbb{Z}$ is an interval. We define the *itinerary map* $\mathcal{I}: T_{\partial D}^1 D \rightarrow X$ as

$$\mathcal{I}_{(x,v)}: \mathbb{Z}_{(x,v)} \rightarrow \{1, 2, \dots, N\}$$

where

$$\begin{cases} (y, w) = f^n(x, v), \\ y \in \gamma_{\mathcal{I}_{(x,v)}(n)}, \end{cases}$$

for all $n \in \mathbb{Z}_{(x,v)}$.

4.1.3 Unfolding

The unfolding technique is standard in the study of polygonal billiards and it consists in reflecting the polygon instead of the trajectory. In this way the motion continues in a smooth way.

Given a polygon D we consider its double billiard surface S and define $S^* = S \setminus V$ (remove the vertices). The *unfolding* of the billiard is the universal covering \tilde{S} of S^* . The unfolding of a trajectory of a point is the lifting to \tilde{S} of the trajectory of x in S by the geodesic flow. See Figure 3.

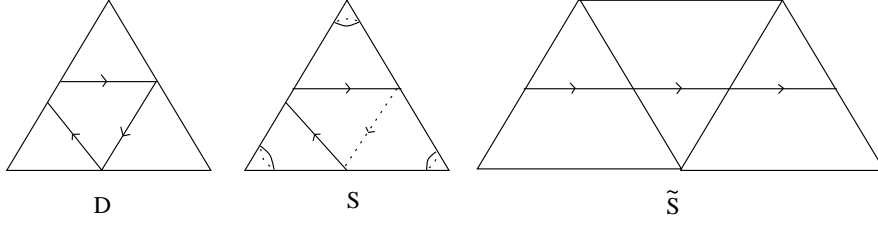


Figure 3: Trajectories in the billiard D , the double surface S and the unfolding surface \tilde{S} .

4.2 Expansive billiard flows

Roughly speaking the expansiveness of the billiard flow can be contradicted just by two facts: 1) there are two points with the same itinerary and 2) there are two points x and y such that $\alpha(x) = \alpha(y) = \{p\}$ and $\omega(x) = \omega(y) = \{q\}$ with p and q singular points. So, we will analyze this possibilities according to the sign of the curvature of the billiard.

4.2.1 Negative curvature

Theorem 3. *If D is a polygon on the Poincaré disc then the following statements hold:*

1. *the itinerary map is injective,*
2. *the billiard flow is expansive and*
3. *the billiard flow has a dense set of periodic orbits.*

Proof. (1) To show that the itinerary map is injective we consider the unfolding billiard surface \tilde{S} defined above. Suppose that x, y in \tilde{S} correspond to two different points (p_x, v_x) and (p_y, v_y) with $p_x, p_y \in \partial D$. Then the trajectories of x and y are divergent, for $t \rightarrow +\infty$ or $t \rightarrow -\infty$. Thus the itineraries of the points are different.

(2) Given $\varepsilon > 0$ we will construct an expansive constant $\delta < 0$ for the billiard flow. First consider a regular point $x \in M$ for the billiard flow. Take a local cross section R_x and a flow box $B_x = \phi_{(-t_x, t_x)} R_x$ such that the diameter of every orbit segment contained in B_x is smaller than ε .

Around each singular point p of the flow consider an adapted neighborhood, as in Figure 4, with the same property: the diameter of every regular orbit segment contained in B_x is smaller than ε . Let $r = s$ or u such that $W^r(p)$ has dimension 2. Define D_p as the connected component of $W^r(p) \cap B_p$ that contains p . Let D_p^+ and D_p^- be the local cross sections, contained in the boundary of B_p , given in figure 4. Let B_p^+ and B_p^- the connected components of $B_p \setminus D_p$ such that $D_p^+ \subset \partial B_p^+$ and $D_p^- \subset \partial B_p^-$, as in Figure 4. Define

$$\delta_p = \min\{\text{dist}(D_p^+, B_p^-), \text{dist}(D_p^-, B_p^+)\}.$$

Since M is compact we can take a finite covering of M of the form

$$\mathcal{B} = \{B_{x_1}, \dots, B_{x_n}, B_{p_1}, \dots, B_{p_k}\},$$

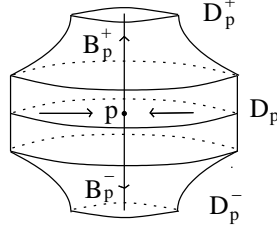


Figure 4: Adepted neighborhood of a singular point p .

being x_1, \dots, x_n some regular points and $\text{Sing} = \{p_1, \dots, p_k\}$.

Consider $\delta \in (0, \min\{\delta_p : p \in \text{Sing}\})$ such that if $x, y \in M$ and $\text{dist}(x, y) < \delta$ then there is $B \in \mathcal{B}$ containing x and y .

Now we will show that δ is an expansive constant for the billiard flow. By contradiction assume that $x, y \in M$ are not contained in an orbit segment of diameter ε and there exists a reparametrization h such that $\text{dist}(\phi_t x, \phi_{h(t)} y) < \delta$ for all $t \in \mathbb{R}$.

First suppose that the ω -limit set of x and y are singular points. It is easy to see that in fact this singular points must be the same point $p \in \text{Sing}$. Since there are no conjugated points in the Poincaré disc we have that the α -limit set of x and y are not singular points. So, eventually changing ϕ_t by ϕ_{-t} , we assume that neither $\omega(x)$ nor $\omega(y)$ are singular points.

Consider $\pi : M \rightarrow S$ the projection on the double surface S . Now lift $\pi(x)$ and $\pi(y)$ to two close points \tilde{x} and \tilde{y} in the unfolding surface \tilde{S} . The trajectory of \tilde{x} can intersect the trajectory of \tilde{y} in at most one point since there are no conjugated points. So eventually changing x and y by $\phi_t x$ and $\phi_{h(t)} y$ respectively, for some $t > 0$, we can suppose that the positive trajectories of \tilde{x} and \tilde{y} do not intersect in \tilde{S} .

Consider a small arc $\gamma \subset \tilde{S}$ connecting \tilde{x} and \tilde{y} . Since the itinerary is injective there is a point $\tilde{z} \in \gamma$ such that $\omega(\tilde{z})$ is a vertex and the positive trajectory of \tilde{z} is contained in the region of \tilde{S} bounded by γ and the positive trajectories of \tilde{x} and \tilde{y} . See figure 5. Now we view the points in M . So there

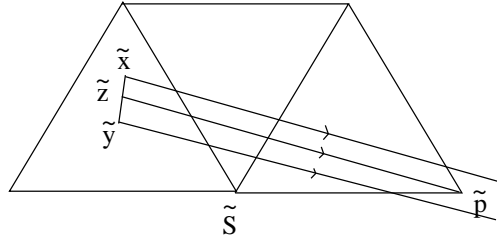


Figure 5: Unfolding the trajectories of x, y and z .

is t such that $\phi_t x \in B_p^-$ (or B_p^+) and $\phi_{h(t)} y \in B_p^+$ (resp. B_p^-). Then consider $s > t$ such that $\phi_{h(s)} y \in D_p^+$ and $\phi_s x \in B_p^-$ (or $\phi_{h(s)} y \in B_p^+$ and $\phi_s x \in D_p^-$). See figure 6.

But it is a contradiction by the way we choose the expansive constant δ .

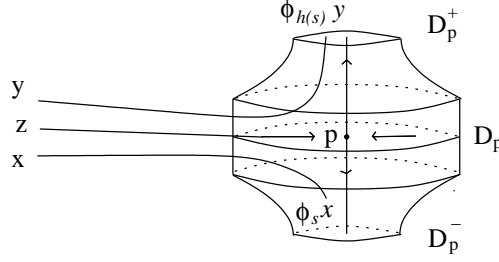


Figure 6: The trajectories in M .

(3) In [4] it is shown (see page 83) that the collision map of a geodesic polygon in the hyperbolic plane has hyperbolic dynamics, that is, has nonvanishing Lyapunov exponents. So we can apply the results in [7] (see Theorem 13.2 on page 155) to conclude that there is a dense set of periodic points. The details of the proof of the cited result are in [6]. \square

Example 3. Consider a simply connected polygon D . Then by Corollary 1 we have that $\pi_1(M)$ has not exponential growth and by the previous Theorem the flow is expansive. It shows that the results of [9] do not extend to expansive flows with singular points.

4.2.2 Flat tables

The following Theorem is similar to the results in [3] and [1].

Theorem 4. If $k = 0$ then the following statements are equivalent:

1. the itinerary map is injective,
2. the billiard flow is expansive and
3. the billiard flow has no periodic orbits.

Proof. (1 \rightarrow 2). The proof of item (2) in Theorem 3 holds in this case because we are assuming that the itinerary is injective and in the flat case as in the hyperbolic case there are no conjugated points.

(2 \rightarrow 3). In the case of flat billiards it is easy to see that if there is a periodic point (x, v) then every initial condition (y, w) , with y sufficiently close to x and w parallel to v , is also periodic. So the flow can not be expansive.

(3 \rightarrow 1). This follows by Theorem 2 in [3]. \square

Problem 1. Does there exists flat polygonal billiards without periodic orbits?

The previous Theorem states this problem in the context of expansive flows with singular points on closed three dimensional manifolds. In particular, if the polygon is a triangle it is a problem on the sphere S^3 .

4.2.3 Spherical billiards

A *generalized diagonal* of D is a billiard trajectory that connects two vertices. Two vertices are *conjugated* if there is a generalized diagonal from one of them to the other with length equals to $k\pi$ for some $k = 1, 2, 3, \dots$

Theorem 5. *If the curvature of the billiard is 1 then the following statements hold:*

1. *If the billiard flow has peridodic orbits then it is not expansive.*
2. *If D has no conjugated vertices and the itinerary map is injective then the billiard flow is expansive.*
3. *If the billiard flow is expansive then the itinerary map is injective.*

Proof. (1) In spherical billiards, if there is a periodic orbit then arbitrary close trajectories contradicts expansiveness for arbitrary small expansive constants. It can be seen by the unfolding technique.

(2) It follows by the arguments of Item (2) in Theorem 3.

(3) Assume by contradiction that there are two different trajectories with the same itinerary. Then unfolding the orbits it is easy to see that the points between them contradicts expansiveness for arbitrary small expansive constants. \square

Now we give an example showing that there are triangular billiards on the sphere S^2 without periodic orbits.

Example 4. *Let $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ be the two dimensional sphere with constant curvature $k = 1$. Consider the triangle $D \subset S^2$ with vertices $V_1 = (0, 0, 1)$, $V_2 = (1, 0, 0)$ and $V_3 = (\cos \theta, \sin \theta, 0)$ being θ an irrational angle. See figure 7.*

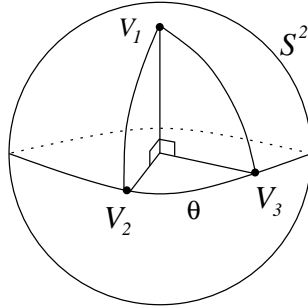


Figure 7: A billiard on the sphere without periodic orbits if θ is irrational.

Notice that every billiard trajectory on D hits infinite times the boundary V_2V_3 . So we do not loose generality supposing that the trajectory starts in the side V_2V_3 . Consider an initial condition (x, v) with $x = (\cos \alpha, \sin \alpha, 0)$, the direction of v is arbitrary. Now with the help of the unfolding technique (see figure 8) it is easy to see that the trajectory is periodic if and only if $\alpha = k\pi + \alpha + 2n\theta$ for some $k, n \in \mathbb{Z}$. But this is impossible if θ is irrational. Therefore D has no periodic orbits.

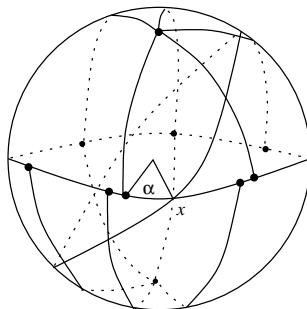


Figure 8: The unfolding of a triangle on the sphere.

Also, the billiard flow of D is not expansive. Again, by the unfolding, it is easy to see that two trajectories starting (x, v) and (x, v') with $x \in V_1 V_3$ are close for all $t \in \mathbb{R}$ if v and v' are close (the trajectories will have the same itinerary). This example also shows that the result of [3] do not hold for spherical billiards: this example has different orbits with the same itinerary but they are not periodic.

Remark 1. The previous example has conjugated vertices, but the flow is not expansive and the itinerary map is not injective. It is obvious that the existence of conjugated vertices prevents the flow of being expansive.

Problem 2. Does there exists polygonal billiards on the sphere with expansive billiard flow?

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